Answer **all** questions in the answer booklet provided. Please start each question on a new page. Full marks are not necessarily awarded for a correct answer with no working. Answers must be supported by working and/or explanations. Solutions found from a graphic display calculator should be supported by suitable working. For example, if graphs are used to find a solution, you should sketch these as part of your answer. Where an answer is incorrect, some marks may be given for a correct method, provided this is shown by written working. You are therefore advised to show all working.

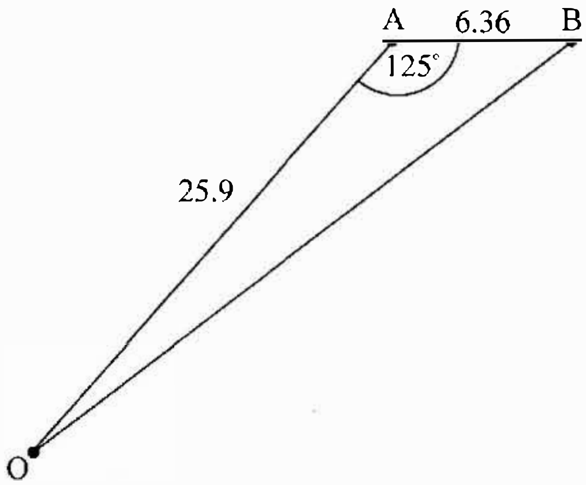
1. [Maximum mark: 13]

The diagram shows points in a park viewed from above, at a specific moment in time.

The distance between two trees, at points A and B, is $6.36\,\mathrm{m}$.

Odette is playing football in the park and is standing at point O, such that $\mathrm{OA} = 25.9\,\mathrm{m}$ and $\mathrm{O\hat{A}B} = 125^\circ$.

diagram not to scale



(a) Calculate the area of triangle AOB. [3]

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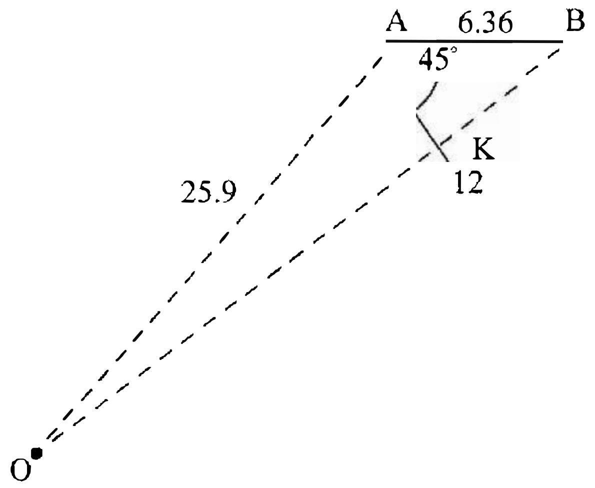
(Question 1 continued)

Odette's friend, Khemil, is standing at point K such that he is $12\,m$ from A and $K\hat{A}B = 45^\circ$.

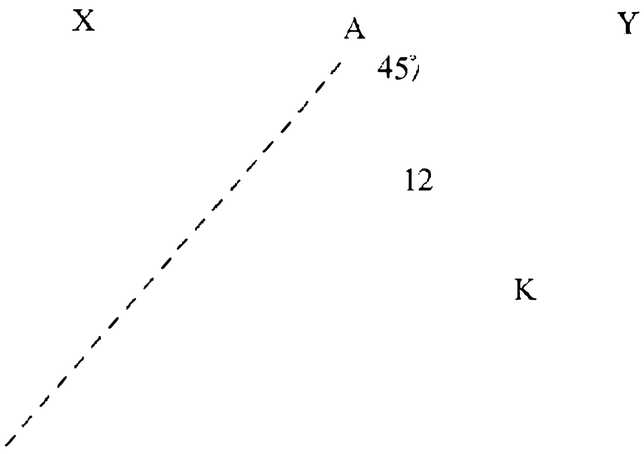
diagram not to scale



(b) Calculate Khemil's distance from B. [3]

XY is a semicircular path in the park with centre A, such that $K\hat{A}Y = 45^\circ$. Khemil is standing on the path and Odette's football is at point X. This is shown in the diagram below.

diagram not to scale



The length $KX = 22.2\,m$, $K\hat{O}X = 53.8^\circ$ and $O\hat{K}X = 51.1^\circ$.

(c) Find whether Odette or Khemil is closer to the football. [4]

Khemil runs along the semicircular path to pick up the football.

(d) Calculate the distance that Khemil runs. [3]

2. [Maximum mark: 12]

A scientist is conducting an experiment on the growth of a certain species of bacteria.

The population of the bacteria, P, can be modelled by the function

$$P(t) = 1200 \times k^t,\ t \geq 0,$$

where t is the number of hours since the experiment began, and k is a positive constant.

(a) (i) Write down the value of $P(0)$.

(ii) Interpret what this value means in this context. [2]

3 hours after the experiment began, the population of the bacteria is $18\,750$.

(b) Find the value of k. [2]

(c) Find the population of the bacteria 1 hour and 30 minutes after the experiment began. [2]

The scientist conducts a second experiment with a different species of bacteria.
The population of this bacteria, S, can be modelled by the function

$$S(t) = 5000 \times 1.65^t,\ t \geq 0,$$

where t is the number of hours since both experiments began.

(d) Find the value of t when the two populations of bacteria are equal. [2]

It takes 2 hours and m minutes for the number of bacteria in the second experiment to reach $19\,000$.

(e) Find the value of m, giving your answer as an integer value. [4]

3. [Maximum mark: 16]

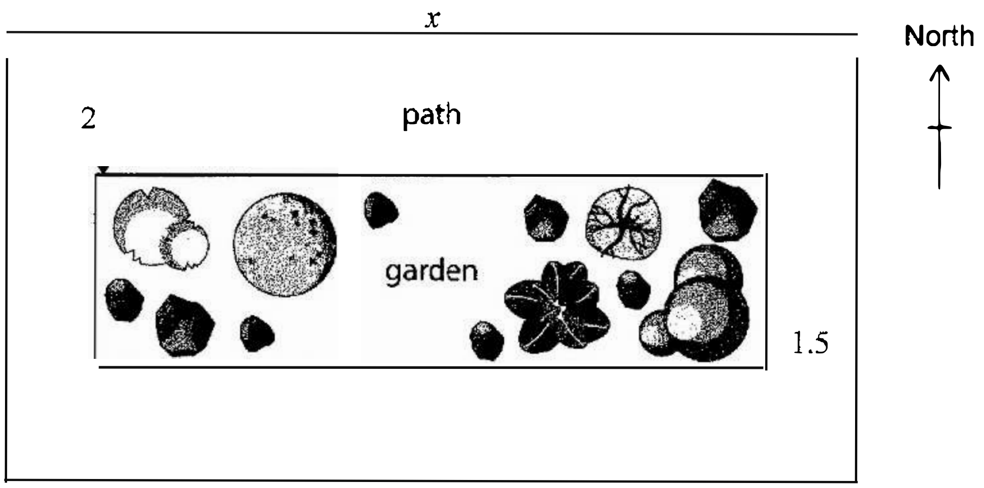
A particular park consists of a rectangular garden, of area $A\,\text{m}^2$, and a concrete path surrounding it. The park has a total area of $1200\,\text{m}^2$.

The width of the path at the north and south side of the park is $2\,\text{m}$.

The width of the path at the west and east side of the park is $1.5\,\text{m}$.

The length of the park (along the north and south sides) is x metres, $3 < x < 300$.

diagram not to scale



(a) Show that $A = 1212 - 4x - \dfrac{3600}{x}$. [5]

(b) Find the possible dimensions of the park if the area of the garden is $800\,\text{m}^2$. [4]

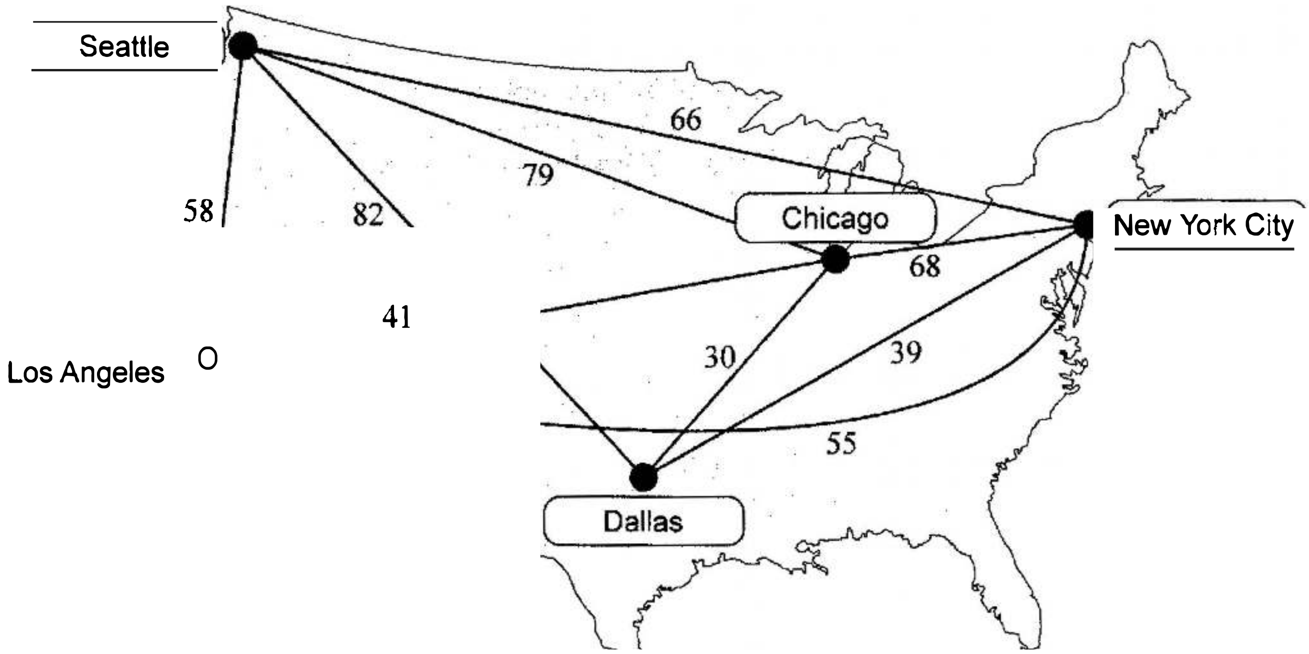
(c) Find an expression for $\dfrac{\mathrm{d}A}{\mathrm{d}x}$. [3]

(d) Use your answer from part (c) to find the value of x that will maximize the area of the garden. [2]

(e) Find the maximum possible area of the garden. [2]

4. [Maximum mark: 19]

The following graph shows five cities of the USA connected by weighted edges representing the cheapest direct flights in dollars ($) between cities.



(a) Explain why the graph can be described as "connected", but not "complete". [2]

(b) Find a minimum spanning tree for the graph using Kruskal's algorithm.

State clearly the order in which your edges are added, and draw the tree obtained. [3]

(c) Using only the edges obtained in your answer to part (b), find an upper bound for the travelling salesman problem. [2]

Ronald lives in New York City and wishes to fly to each of the other cities, before finally returning to New York City. After some research, he finds that there exists a direct flight between Los Angeles and Dallas costing $\$26$. He updates the graph to show this.

(d) By using the nearest neighbour algorithm and starting at Los Angeles, determine a better upper bound than that found in part (c).

State clearly the order in which you are adding the vertices. [3]

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(Question 4 continued)

(e) (i) By deleting the vertex which represents Chicago, use the deleted vertex algorithm to determine a lower bound for the travelling salesman problem.

(ii) Similarly, by instead deleting the vertex which represents Seattle, determine another lower bound. [5]

(f) Hence, using your previous answers, write down your best inequality for the **least** expensive tour Ronald could take. Let the variable C represent the total cost, in dollars, for the tour. [2]

(g) Write down a tour that is strictly greater than your lower bound and strictly less than your upper bound. [2]

5. [Maximum mark: 14]

The three countries of Belgium, Germany and The Netherlands meet at a single point called Vaalserberg.

To support future transport planning, a 10 km circle was drawn around Vaalserberg on a map. A study was carried out over five years to determine what percentage of people living in each of these countries (within the 10 km circular region) either stayed in their own country or moved to another country within the circle.

From this study, the following movements were observed during the five years.

- From Belgium, 5% moved to Germany, and 0.5% moved to The Netherlands.
- From Germany, 2% moved to The Netherlands, and 1.5% moved to Belgium.
- From The Netherlands, 3% moved to Germany, and 2% moved to Belgium.

All additional population changes within the circular region may be ignored.

(a) Represent the above information in a transition matrix $\boldsymbol{T}$. [3]

At the end of the study, the population of the Belgian side was $26\,000$, the population of the German side was $240\,000$, and the population of The Netherlands side was $50\,000$.

(b) By using $\boldsymbol{T}$, find the expected population of the German side of Vaalserberg 30 years after the end of the study. [4]

For matrix $\boldsymbol{T}$ there exists a steady state vector

$$\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix},$$

where u_1, u_2 and u_3 are the proportions of the total population on the Belgian side, the German side and The Netherlands side respectively.

The steady state vector $\boldsymbol{u}$ may be found by solving a system of equations.

(c) (i) Determine these equations that are to be solved.

(ii) By solving your system of equations, find $\boldsymbol{u}$. [3]

(d) Use your answer to part (c)(ii) to determine the long-term expected population of the German side. [2]

(e) Suggest two reasons why your answer to part (d) is not likely to be accurate. You may comment on both the model and the situation in context. [2]

6. [Maximum mark: 18]

The gardener in a local park suggested that the number of snails found in the park can be modelled by a Poisson distribution.

(a) Suggest two observations that the gardener may have made that led him to suggest this model. [2]

Now assume that the model is valid and that the mean number of snails per m^2 is 0.2. The gardener inspects, at random, a $12\,\mathrm{m}^2$ area of the park.

(b) Find the probability that the gardener finds exactly four snails. [3]

(c) Find the probability that the gardener finds fewer than three snails. [2]

(d) Find the probability that, in three consecutive inspections, the gardener finds at least one snail per inspection. [3]

Following heavy rain overnight, the gardener wished to determine whether the number of snails found in a random $12\,\mathrm{m}^2$ area of the park had increased.

(e) State the hypotheses for the test. [2]

(f) Find the critical region for the test at the 1% significance level. [3]

(g) Given that the mean number of snails per m^2 has actually risen to 0.75, find the probability that the gardener makes a Type II error. [3]

7. [Maximum mark: 18]

A biologist suggests that the rates of change of the population of fruit flies (after time $t \geq 0$) in a particular ecosystem are given by the following equations, where x is the population of male fruit flies and y is the population of female fruit flies.

$$\frac{\mathrm{d}x}{\mathrm{d}t} = -4x + 6y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 9x - y$$

(a) Find the eigenvalues and corresponding eigenvectors of the matrix $\begin{pmatrix} -4 & 6 \\ 9 & -1 \end{pmatrix}$. [6]

(b) Hence write down the general solution of the system, giving your answer in the form $\begin{pmatrix} x \\ y \end{pmatrix} = A\boldsymbol{p}_1 \mathrm{e}^{\lambda_1 t} + B\boldsymbol{p}_2 \mathrm{e}^{\lambda_2 t}$, where A, B, λ_1, λ_2 $(\lambda_2 > \lambda_1)$ are scalar constants and $\boldsymbol{p}_1$, $\boldsymbol{p}_2$ are vector constants. [2]

Initially $x = 500$ and $y = 125$.

(c) Determine the value of A and the value of B. [2]

(d) State the long-term ratio of male fruit flies to female fruit flies. [1]

(e) Find the value of $\mathrm{d}y$ at time $t = 0$. [3]

(f) Sketch the trajectory, on the phase portrait, for the population growth of the fruit flies. [4]